

# Two positivity conjectures for Kerov polynomials

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## Abstract

Kerov polynomials express the normalized characters of irreducible representations of the symmetric group, evaluated on a cycle, as polynomials in the “free cumulants” of the associated Young diagram. We present two positivity conjectures for their coefficients. The latter are stronger than the positivity conjecture of Kerov-Biane, recently proved by Féray.

## 1 Kerov polynomials

### 1.1 Characters

A partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  is a finite weakly decreasing sequence of nonnegative integers, called parts. The number  $l(\lambda)$  of positive parts is called the length of  $\lambda$ , and  $|\lambda| = \sum_{i=1}^r \lambda_i$  the weight of  $\lambda$ . For any integer  $i \geq 1$ ,  $m_i(\lambda) = \text{card}\{j : \lambda_j = i\}$  is the multiplicity of the part  $i$  in  $\lambda$ .

Let  $n$  be a fixed positive integer and  $S_n$  the group of permutations of  $n$  letters. Each permutation  $\sigma \in S_n$  factorizes uniquely as a product of disjoint cycles, whose respective lengths are ordered such as to form a partition  $\mu = (\mu_1, \dots, \mu_r)$  with weight  $n$ , the so-called cycle-type of  $\sigma$ .

The irreducible representations of  $S_n$  and their corresponding characters are also labelled by partitions  $\lambda$  with weight  $|\lambda| = n$ . We write  $\dim \lambda$  for the dimension of the representation  $\lambda$  and  $\chi_\mu^\lambda$  for the value of the character  $\chi^\lambda(\sigma)$  at any permutation  $\sigma$  of cycle-type  $\mu$ .

Let  $r \leq n$  be a positive integer and  $\mu = (r, 1^{n-r})$  the corresponding  $r$ -cycle in  $S_n$ . We write

$$\hat{\chi}_r^\lambda = n(n-1) \cdots (n-r+1) \frac{\chi_{r, 1^{n-r}}^\lambda}{\dim \lambda}$$

for the value at  $\mu$  of the normalized character.

It was first observed by Kerov[6] and Biane[2] that  $\hat{\chi}_r^\lambda$  may be written as a polynomial in the “free cumulants” of the Young diagram of  $\lambda$ .

## 1.2 Free cumulants

Two increasing sequences  $y = (y_1, \dots, y_{d-1})$  and  $x = (x_1, \dots, x_{d-1}, x_d)$  are said to be interlacing if  $x_1 < y_1 < x_2 < \dots < x_{d-1} < y_{d-1} < x_d$ . The center of the pair is  $c(x, y) = \sum_i x_i - \sum_i y_i$ .

To any pair of interlacing sequences with center 0 we associate the rational function

$$G_{x,y}(z) = \frac{1}{z - x_d} \prod_{i=1}^{d-1} \frac{z - y_i}{z - x_i},$$

and the formal power series inverse to  $G_{x,y}$  for composition,

$$G_{x,y}^{(-1)}(z) = z^{-1} + \sum_{k \geq 1} R_k(x, y) z^{k-1}.$$

Note that  $R_1(x, y) = c(x, y) = 0$ . The quantities  $R_k(x, y), k \geq 2$  are called the free cumulants of the interlacing pair  $(x, y)$ .

Being given a partition  $\lambda$ , we consider the collection of unit boxes centered on the nodes  $\{(j - 1/2, i - 1/2) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$ . This defines a compact region in  $\mathbb{R}^2$ , the so-called Young diagram of  $\lambda$ . On  $\mathbb{R}^2$  we define the content function by  $c(u, v) = u - v$ . By convention, the content of a box is the one of its center.

Then it is easily shown that the Young diagram of  $\lambda$  defines a pair of interlacing sequences, formed by the contents  $y_1, \dots, y_{d-1}$  of its corner boxes, and the contents  $x_1, \dots, x_{d-1}, x_d$  of the corner boxes of its compliment in  $\mathbb{R}^2$ . We have  $x_1 = -l(\lambda)$ , and  $x_d = \lambda_1$ .

Conversely, every pair of interlacing sequences with integer entries and center zero uniquely determines the Young diagram of a partition  $\lambda$ .

The free cumulants  $R_k(\lambda), k \geq 2$  are defined accordingly. These quantities arise in the asymptotic study of representations of symmetric groups [1].

## 1.3 Known results

The following result was first proved in [2] and attributed to Kerov[6].

**Theorem.** *There exist polynomials  $K_r, r \geq 2$  such that for any partition  $\lambda$  with  $|\lambda| \geq r$ , one has*

$$\hat{\chi}_r^\lambda = K_r(R_2(\lambda), R_3(\lambda), \dots, R_{r+1}(\lambda)).$$

*These polynomials have integer coefficients.*

Let  $(R_2, \dots, R_{r+1})$  be the indeterminates of the “Kerov polynomial”  $K_r$  and define  $|\mu|$  as the “weight” of the monomial  $R_\mu = \prod_{i \geq 2} R_i^{m_i(\mu)}$ . We may decompose  $K_r$  in its graded components with respect to the weight, writing

$$K_r = \sum_{s \geq 2} K_{r,s} \quad \text{with} \quad K_{r,s} = \sum_{|\mu|=s} x_\mu^{(r)} \prod_{i \geq 2} R_i^{m_i(\mu)}.$$

Then it may be proved [2] that the term of highest weight is  $R_{r+1}$  and that  $K_{r,s} = 0$  when  $s = r - 2k$ .

Goulden and Rattan [5, 10] have given a general formula for  $K_{r,r-2k+1}$ , expressing it as some coefficient in a formal power series (see also [3]). As a consequence, one has

$$K_{r,r-1} = \frac{1}{4} \binom{r+1}{3} \sum_{|\mu|=r-1} l(\mu)! \prod_{i \geq 2} \frac{((i-1)R_i)^{m_i(\mu)}}{m_i(\mu)!},$$

which had been conjectured by Biane [2] and differently proved by Śniady [11].

The same method provides an explicit form for  $K_{r,r-3}$ . But as far as  $K_{r,r-5}$  (and lower components) are concerned, it seems very difficult to apply. Rattan [10, Theorem 3.5.12] found a messy expression of  $K_{r,r-5}$  giving an idea about the complexity of the problem.

The following positivity property had been conjectured by Kerov [6] and Biane [2] and was recently proved by Féray [4].

**Theorem.** *The coefficients of  $K_r$  are nonnegative integers.*

The purpose of this note is to present a stronger conjectural property.

## 2 Conjectures

An algebraic basis of the (abstract) symmetric algebra with real coefficients is formed by the classical symmetric functions, elementary  $e_i$ , complete  $h_i$  or power-sum  $p_i$ . As usual for any partition  $\mu$ , denote  $e_\mu$ ,  $h_\mu$  or  $p_\mu$  their product over the parts of  $\mu$ , and  $m_\mu$  the monomial symmetric function, sum of all distinct monomials whose exponent is a permutation of  $\mu$ .

For a clearer display we write

$$\mathcal{R}_\mu = \prod_{i \geq 2} ((i-1)R_i)^{m_i(\mu)} / m_i(\mu)!.$$

Firstly we conjecture that the Kerov components  $K_{r,r-2k+1}$  may be described in a unified way, *independent of  $r$* .

**Conjecture 1.** *For any  $k \geq 1$  there exists an inhomogeneous symmetric function  $f_k$ , having maximal degree  $4(k-1)$ , such that*

$$K_{r,r-2k+1} = \binom{r+1}{3} \sum_{|\mu|=r-2k+1} (l(\mu) + 2k - 2)! f_k(\mu) \mathcal{R}_\mu,$$

where  $f_k(\mu)$  denotes the value of  $f_k$  at the integral vector  $\mu$ . This symmetric function is *independent of  $r$* .

The assertion is trivial for  $k = 1$  since we have  $f_1 = 1/4$ . Secondly we conjecture the symmetric function  $f_k$  to be positive in the following sense.

**Conjecture 2.** For  $k \geq 2$  the inhomogeneous symmetric function  $f_k$  may be written

$$f_k = \sum_{|\rho| \leq 4(k-1)} c_\rho^{(k)} m_\rho,$$

where the coefficients  $c_\rho^{(k)}$  are positive rational numbers.

The positivity of the coefficients of  $K_{r,r-2k+1}$  is an obvious consequence. We emphasize that the coefficients of  $f_k$  in terms of any other classical basis *may be negative*.

Conjecture 2 is firstly supported by the case  $k = 2$ . Using the expression of  $K_{r,r-3}$  given in [5], we have the following result, whose proof is postponed to Section 3.

**Theorem 1.** For  $k = 2$ , we have

$$5760f_2 = 3m_4 + 8m_{31} + 10m_{22} + 16m_{21^2} + 24m_{1^4} \\ + 20m_3 + 36m_{21} + 48m_{1^3} + 35m_2 + 40m_{1^2} + 18m_1.$$

Conjecture 2 is secondly supported by extensive computer calculations, giving the values of the positive numbers  $c_\rho^{(k)}$  for  $k = 3, 4$ . The two following conjectures have been checked for any  $K_r$  with  $r \leq 32$ .

**Conjecture 3.** For  $k = 3$ , the values of  $2.6!.8!c_\rho^{(3)}$  are given by the table below.

8	71	62	61 <sup>2</sup>	53	521	51 <sup>3</sup>	4 <sup>2</sup>	431	42 <sup>2</sup>	421 <sup>2</sup>
9	48	132	224	240	544	908	294	848	1132	1904
41 <sup>4</sup>	3 <sup>2</sup> 2	3 <sup>2</sup> 1 <sup>2</sup>	32 <sup>2</sup> 1	321 <sup>3</sup>	31 <sup>5</sup>	2 <sup>4</sup>	2 <sup>3</sup> 1 <sup>2</sup>	2 <sup>2</sup> 1 <sup>4</sup>	21 <sup>6</sup>	1 <sup>8</sup>
3148	1440	2440	3280	5480	9040	4440	7440	12360	20400	33600
7	61	52	51 <sup>2</sup>	43	421	41 <sup>3</sup>	3 <sup>2</sup> 1	32 <sup>2</sup>	321 <sup>2</sup>	31 <sup>4</sup>
216	968	2296	3744	3560	7704	12368	9856	13072	21264	33968
2 <sup>3</sup> 1	2 <sup>2</sup> 1 <sup>3</sup>	21 <sup>5</sup>	1 <sup>7</sup>							
28560	46080	73680	117600							
6	51	42	41 <sup>2</sup>	3 <sup>2</sup>	321	31 <sup>3</sup>	2 <sup>3</sup>	2 <sup>2</sup> 1 <sup>2</sup>	21 <sup>4</sup>	1 <sup>6</sup>
2094	7696	15450	24016	19696	40592	62428	53796	83848	128988	198120
5	41	32	31 <sup>2</sup>	2 <sup>2</sup> 1	21 <sup>3</sup>	1 <sup>5</sup>				
10588	30972	51096	75232	99640	146200	214040				
4	31	2 <sup>2</sup>	21 <sup>2</sup>	1 <sup>4</sup>						
30109	67360	87382	120912	166320						
3	21	1 <sup>3</sup>								
48092	77684	98016								
2	1 <sup>2</sup>	1								
39884	43928	13200								

**Conjecture 4.** For  $k = 4$ , the values of  $2.8!.12!c_p^{(4)}$  are given by the table below.

12	11,1	10,2	$10,1^2$	93	921	$91^3$
495	3960	16830	29040	48312	113520	194392
84	831	$82^2$	$821^2$	$81^4$	75	741
99297	296472	403590	692912	1180248	150480	546480
732	$731^2$	$72^21$	$721^3$	$71^5$	66	651
945120	1626592	2219360	3792480	6439200	172260	733920
642	$641^2$	$63^2$	6321	$631^3$	$62^3$	$62^21^2$
1543740	2654432	1960992	4611552	7890528	6305640	10797440
$621^4$	$61^6$	$5^22$	$5^21^2$	543	5421	$541^3$
18388320	31168800	1811040	3110800	2797872	6566560	11221392
$53^21$	$532^2$	$5321^2$	$531^4$	$52^31$	$52^21^3$	$521^5$
8360352	11420640	19573792	33343968	26812800	45753120	77733600
$51^7$	$4^3$	$4^231$	$4^22^2$	$4^221^2$	$4^21^4$	$43^22$
131644800	3402630	10157840	13861540	23751840	40429200	17629920
$43^21^2$	$432^21$	$4321^3$	$431^5$	$42^4$	$42^31^2$	$42^21^4$
30274720	41416480	70724640	120150240	56773080	97050240	165207840
$421^6$	$41^8$	$3^4$	$3^321$	$3^31^3$	$3^22^3$	$3^22^21^2$
280274400	474445440	22397760	52718400	90162240	72246720	123618880
$3^221^4$	$3^21^6$	$32^41$	$32^31^3$	$32^21^5$	$321^7$	$31^9$
210584640	357315840	169727040	289477440	492072000	834301440	1412328960
$2^6$	$2^51^2$	$2^41^4$	$2^31^6$	$2^21^8$	$21^{10}$	$1^{12}$
233226000	398160000	677678400	1150632000	1950278400	3302208000	5588352000

11	10,1	92	$91^2$	83	821	$81^3$
25740	184140	708444	1199440	1836252	4210844	7075728
74	731	$72^2$	$721^2$	$71^4$	65	641
3371544	9817984	13294160	22416768	37488576	4518360	15961880
632	$631^2$	$62^21$	$621^3$	$61^5$	$5^21$	542
27441744	46345728	62955728	105656256	176236800	18678880	38954344
$541^2$	$53^2$	5321	$531^3$	$52^3$	$52^21^2$	$521^4$
65688480	49454592	113453824	190456128	154321200	259558848	434150016
$51^6$	$4^23$	$4^221$	$4^21^3$	$43^21$	$432^2$	$4321^2$
723211200	59989160	137408040	230425440	174697600	237252400	399329280
$431^4$	$42^31$	$42^21^3$	$421^5$	$41^7$	$3^32$	$3^31^2$
667719360	544244400	912287040	1522644480	2534616000	301150080	507776640
$3^22^21$	$3^221^3$	$3^21^5$	$32^4$	$32^31^2$	$32^21^4$	$321^6$
691290880	1159603200	1935373440	942671520	1583527680	2648849280	4416612480
$31^8$	$2^51$	$2^41^3$	$2^31^5$	$2^21^7$	$21^9$	$1^{11}$
7350658560	2163722400	3624808320	6054048000	10089797760	16795537920	27941760000

10	91	82	81 <sup>2</sup>	73	721
589545	3732696	12880197	21347832	29796624	66542608
71 <sup>3</sup>	64	631	62 <sup>2</sup>	621 <sup>2</sup>	61 <sup>4</sup>
109503504	48249234	136592720	184006988	304004800	498221712
5 <sup>2</sup>	541	532	531 <sup>2</sup>	52 <sup>2</sup> 1	521 <sup>3</sup>
56379312	192905680	329380304	544358320	736055232	1210234416
51 <sup>5</sup>	4 <sup>2</sup> 2	4 <sup>2</sup> 1 <sup>2</sup>	43 <sup>2</sup>	4321	431 <sup>3</sup>
1979174160	398071454	657018384	504522128	1126245296	1850729904
42 <sup>3</sup>	42 <sup>2</sup> 1 <sup>2</sup>	421 <sup>4</sup>	41 <sup>6</sup>	3 <sup>3</sup> 1	3 <sup>2</sup> 2 <sup>2</sup>
1523067348	2510092224	4113855024	6719636880	1427727840	1928190880
3 <sup>2</sup> 21 <sup>2</sup>	3 <sup>2</sup> 1 <sup>4</sup>	32 <sup>3</sup> 1	32 <sup>2</sup> 1 <sup>3</sup>	321 <sup>5</sup>	31 <sup>7</sup>
3180030560	5210415840	4309828320	7080806880	11585384160	18913547520
2 <sup>5</sup>	2 <sup>4</sup> 1 <sup>2</sup>	2 <sup>3</sup> 1 <sup>4</sup>	2 <sup>2</sup> 1 <sup>6</sup>	21 <sup>8</sup>	1 <sup>10</sup>
5844598200	9618960960	15768879840	25780980960	42087911040	68660524800

9	81	72	71 <sup>2</sup>	63	621
7834926	43370910	132689304	214757664	270145656	585908840
61 <sup>3</sup>	54	531	52 <sup>2</sup>	521 <sup>2</sup>	51 <sup>4</sup>
942097728	380072484	1041283232	1395178488	2251722880	3607638624
4 <sup>2</sup> 1	432	431 <sup>2</sup>	42 <sup>2</sup> 1	421 <sup>3</sup>	41 <sup>5</sup>
1253522292	2121348680	3421048224	4600109272	7388286912	11813196960
3 <sup>3</sup>	3 <sup>2</sup> 21	3 <sup>2</sup> 1 <sup>3</sup>	32 <sup>3</sup>	32 <sup>2</sup> 1 <sup>2</sup>	321 <sup>4</sup>
2679266304	5805122752	9318556608	7798935408	12559063744	20114667264
31 <sup>6</sup>	2 <sup>4</sup> 1	2 <sup>3</sup> 1 <sup>3</sup>	2 <sup>2</sup> 1 <sup>5</sup>	21 <sup>7</sup>	1 <sup>9</sup>
32123903040	16915888080	27152536320	43428598080	69327800640	110563004160

8	71	62	61 <sup>2</sup>	53
66992805	319460328	854070228	1345992736	1504935432
521	51 <sup>3</sup>	4 <sup>2</sup>	431	42 <sup>2</sup>
3156966208	4945126296	1806665454	4760982424	6336879340
421 <sup>2</sup>	41 <sup>4</sup>	3 <sup>2</sup> 2	3 <sup>2</sup> 1 <sup>2</sup>	32 <sup>2</sup> 1
9953455776	15535885752	7959879312	12492469616	16671548080
321 <sup>3</sup>	31 <sup>5</sup>	2 <sup>4</sup>	2 <sup>3</sup> 1 <sup>2</sup>	2 <sup>2</sup> 1 <sup>4</sup>
26065233552	40592042160	22229994072	34840460832	54337307568
21 <sup>6</sup>	1 <sup>8</sup>			
84517248240	131257445760			

7	61	52	51 <sup>2</sup>	43
386137224	1557181296	3572220960	5460878192	5341858632
421	41 <sup>3</sup>	3 <sup>2</sup> 1	32 <sup>2</sup>	321 <sup>2</sup>
10769122320	16360041456	13438992512	17722898864	26967001248
31 <sup>4</sup>	2 <sup>3</sup> 1	2 <sup>2</sup> 1 <sup>3</sup>	21 <sup>5</sup>	1 <sup>7</sup>
40796325216	35619645600	53958337440	81409500480	122509104480

6	51	42	41 <sup>2</sup>	3 <sup>2</sup>	321
1527234687	5086528128	9789272361	14430109232	12134469600	23282303088
31 <sup>3</sup>	2 <sup>3</sup>	2 <sup>2</sup> 1 <sup>2</sup>	21 <sup>4</sup>	1 <sup>6</sup>	
34060600640	30307366254	44384647296	64583789280	93548535360	

5	41	32	$31^2$	$2^21$
4133557494	11019741678	17318813292	24369700608	31165644708
$21^3$	$1^5$			
43403668704	59946923520			
4	31	22	$21^2$	$1^4$
7478442180	15298473960	19094031000	25180566840	32685206400
3	21	$1^3$		
8579601096	12733485336	15147277200		
2	$1^2$	1		
5589321408	5773242816	1555424640		

### 3 Proof of Theorem 1

Following [5, 10] we consider the generating series

$$C(z) = \sum_{i \geq 0} C_i z^i = \left(1 - \sum_{i \geq 2} (i-1) R_i z^i\right)^{-1}.$$

By classical methods we have

$$C_n = \sum_{|\mu|=n} l(\mu)! \mathcal{R}_\mu.$$

It may be shown (see a proof in Section 7 below) that if  $\phi$  is a polynomial in  $i$ , there exists a symmetric function  $\hat{\phi}$  such that

$$\sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} \phi(i) C_i C_j C_k = \sum_{|\mu|=n} (l(\mu) + 2)! \hat{\phi}(\mu) \mathcal{R}_\mu,$$

where  $\hat{\phi}(\mu)$  denotes the value of  $\hat{\phi}$  at the integral vector  $\mu$ . For  $\phi(i) = a + bi + ci^2$ , we have

$$\hat{\phi} = a/2 + bn/6 + c(n^2 + p_2)/12.$$

The following explicit form of  $K_{r,r-3}$  was given in [5, Theorem 3.3]

$$K_{r,r-3} = \binom{r+1}{3} \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=r-3}} (a(r) + b(r)i^2) C_i C_j C_k,$$

with

$$a(r) = -\frac{1}{2880}(r-1)(r-3)(r^2 - 4r - 6), \quad b(r) = \frac{1}{480}(2r^2 - 3).$$

As a straightforward consequence, we have

$$K_{r,r-3} = \binom{r+1}{3} \sum_{|\mu|=r-3} (l(\mu) + 2)! f_2(\mu) \mathcal{R}_\mu,$$

with

$$f_2(\mu) = \frac{1}{2}a(r) + \frac{1}{12}b(r)((r-3)^2 + p_2(\mu)).$$

But since  $|\mu| = p_1(\mu) = r - 3$ , this can be rewritten

$$f_2 = \frac{1}{5760} \left( 2p_2p_1^2 + p_1^4 + 12p_2p_1 + 8p_1^3 + 15p_2 + 20p_1^2 + 18p_1 \right).$$

Using for instance ACE [12] we easily obtain

$$f_2 = \frac{1}{5760} \left( 3m_4 + 8m_{31} + 10m_{22} + 16m_{21^2} + 24m_{1^4} \right. \\ \left. + 20m_3 + 36m_{21} + 48m_{1^3} + 35m_2 + 40m_{1^2} + 18m_1 \right). \quad \square$$

Observe that in this particular situation, the coefficients of  $f_2$  in terms of power sums are nonnegative. This property *is not true* for  $K_{r,r-5}$  and lower components.

Starting from [10, Theorem 3.5.12], Conjecture 3 may probably be proved along the same line.

## 4 C-expansion

Goulden and Rattan [5, 10] have considered the expansion of Kerov polynomials in terms of the indeterminates  $C_i$ . They have given the following positivity conjecture, proved for  $k = 1, 2$ , which is stronger than the one of Kerov and Biane.

**Conjecture.** *For  $k \geq 1$  the coefficients of  $K_{r,r-2k+1}$  in terms of the  $C_i$ 's are nonnegative rational numbers.*

In analogy with Section 2 we conjecture that for any  $k \geq 1$  one has

$$K_{r,r-2k+1} = \binom{r+1}{3} \sum_{\substack{\nu \in \mathbb{N}^{2k-1} \\ |\nu| = r-2k+1}} F_k(\nu) \prod_{i=1}^{2k-1} C_{\nu_i},$$

where  $F_k$  is an inhomogeneous symmetric function, having maximal degree  $4(k-1)$  and independent of  $r$ .

This is clear for  $k = 1$  since

$$K_{r,r-1} = \frac{1}{4} \binom{r+1}{3} C_{r-1},$$

hence  $F_1 = 1/4$ . For  $k = 2$  we have seen in Section 3 that

$$F_2(\nu) = a(r) + \frac{1}{3}b(r)p_2(\nu)$$



with  $\nu = (i, j, k)$ . Since  $|\nu| = p_1(\nu) = r - 3$ , we obtain

$$F_2 = \frac{1}{2880} \left( 4p_2p_1^2 - p_1^4 + 24p_2p_1 - 4p_1^3 + 30p_2 + 5p_1^2 + 18p_1 \right).$$

However we emphasize that, unlike those of  $f_2$ , the coefficients of  $F_2$  in terms of monomial symmetric functions *are not positive*. One has

$$F_2 = \frac{1}{2880} \left( 3m_4 + 4m_{31} + 2m_{22} - 4m_{21^2} + 20m_3 + 12m_{21} - 24m_{1^3} + 35m_2 + 10m_{1^2} + 18m_1 \right).$$

Therefore it seems that  $C$ -positivity and  $R$ -positivity are of a different nature.

## 5 New expansion

For a better understanding of the difference between the  $C$  and  $R$  expansions, it is useful to introduce new polynomials  $Q_i$  in the free cumulants. Define  $Q_0 = 1$ ,  $Q_1 = 0$  and for any  $n \geq 2$ ,

$$Q_n = \sum_{|\mu|=n} (l(\mu) - 1)! \mathcal{R}_\mu.$$

Writing for short

$$\mathcal{Q}_\mu = \prod_{i \geq 2} Q_i^{m_i(\mu)} / m_i(\mu)!, \quad \mathcal{C}_\mu = \prod_{i \geq 2} C_i^{m_i(\mu)} / m_i(\mu)!,$$

the correspondence between these three families is given by

$$\begin{aligned} Q_n &= \sum_{|\mu|=n} (-1)^{l(\mu)} (l(\mu) - 1)! \mathcal{C}_\mu, \\ C_n &= \sum_{|\mu|=n} l(\mu)! \mathcal{R}_\mu = \sum_{|\mu|=n} \mathcal{Q}_\mu, \\ (1 - n)R_n &= \sum_{|\mu|=n} (-1)^{l(\mu)} \mathcal{Q}_\mu = \sum_{|\mu|=n} (-1)^{l(\mu)} l(\mu)! \mathcal{C}_\mu. \end{aligned}$$

These relations are better understood by using the theory of symmetric functions. Actually let  $\mathbf{A}$  be the (formal) alphabet defined by

$$(i - 1)R_i = -h_i(\mathbf{A}), \quad Q_i = -p_i(\mathbf{A})/i, \quad C_i = (-1)^i e_i(\mathbf{A}).$$

Writing

$$u_\mu = l(\mu)! / \prod_{i \geq 1} m_i(\mu)!, \quad \epsilon_\mu = (-1)^{n-l(\mu)}, \quad z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!,$$

the previous relations are merely the classical properties [9, pp. 25 and 33]

$$\begin{aligned} p_n &= -n \sum_{|\mu|=n} (-1)^{l(\mu)} u_\mu h_\mu / l(\mu) = -n \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu / l(\mu), \\ e_n &= \sum_{|\mu|=n} \epsilon_\mu u_\mu h_\mu = \sum_{|\mu|=n} \epsilon_\mu z_\mu^{-1} p_\mu, \\ h_n &= \sum_{|\mu|=n} z_\mu^{-1} p_\mu = \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu. \end{aligned}$$

From these relations, it is clear that  $C$ -positivity implies  $Q$ -positivity, which itself implies  $R$ -positivity. In particular the following conjecture is *a priori* stronger than the one of Kerov-Biane and weaker than the one of Goulden-Rattan.

**Conjecture 5.** *For  $k \geq 1$  the coefficients of  $K_{r,r-2k+1}$  in terms of the  $Q_i$ 's are nonnegative rational numbers.*

The assertion is trivial for  $k = 1$  since

$$K_{r,r-1} = \frac{1}{4} \binom{r+1}{3} C_{r-1} = \frac{1}{4} \binom{r+1}{3} \sum_{|\mu|=r-1} \mathcal{Q}_\mu.$$

This leads us to the following conjecture (with obviously  $g_1 = 1/4$ ).

**Conjecture 6.** *For any  $k \geq 1$  there exists an inhomogeneous symmetric function  $g_k$ , having maximal degree  $4(k-1)$ , such that*

$$K_{r,r-2k+1} = \binom{r+1}{3} \sum_{|\mu|=r-2k+1} (2k-1)^{l(\mu)} g_k(\mu) \mathcal{Q}_\mu,$$

where  $g_k(\mu)$  denotes the value of  $g_k$  at the integral vector  $\mu$ . This symmetric function is independent of  $r$ .

It is a highly remarkable fact that, in contrast with the  $C$ -expansion, the  $Q$ -positivity is completely analogous to the  $R$ -positivity (and possibly equivalent).

**Conjecture 7.** *For  $k \geq 2$  the inhomogeneous symmetric function  $g_k$  may be written*

$$g_k = \sum_{|\rho| \leq 4(k-1)} a_\rho^{(k)} m_\rho,$$

where the coefficients  $a_\rho^{(k)}$  are positive rational numbers.

The assertion of Conjecture 5 is a direct consequence. Conjecture 7 is supported by the following result for  $k = 2$ , which will be proved in Section 6.

**Theorem 2.** For  $k = 2$ , we have

$$8640 g_2 = 9m_4 + 20m_{31} + 22m_{22} + 28m_{21^2} + 24m_{1^4} \\ + 60m_3 + 84m_{21} + 72m_{1^3} + 105m_2 + 90m_{1^2} + 54m_1.$$

Conjecture 7 is also supported by computer calculations, giving the positive numbers  $a_\rho^{(k)}$  for  $k = 3, 4$ .

**Conjecture 8.** For  $k = 3$ , the values of  $500.5! \cdot 7! a_\rho^{(3)}$  are given by the table below.

8	71	62	61 <sup>2</sup>	53	521	51 <sup>3</sup>	4 <sup>2</sup>	431	42 <sup>2</sup>
1125	5400	13500	21480	23400	46200	69072	28350	69000	84900
421 <sup>2</sup>	41 <sup>4</sup>	3 <sup>2</sup> 2	3 <sup>2</sup> 1 <sup>2</sup>	32 <sup>2</sup> 1	321 <sup>3</sup>	31 <sup>5</sup>	2 <sup>4</sup>	2 <sup>3</sup> 1 <sup>2</sup>	2 <sup>2</sup> 1 <sup>4</sup>
126168	174864	104400	157152	190704	265632	338880	233208	322128	414432
21 <sup>6</sup>	1 <sup>8</sup>								
486720	524160								
7	61	52	51 <sup>2</sup>	43	421	41 <sup>3</sup>	3 <sup>2</sup> 1	32 <sup>2</sup>	
27000	107400	231000	345360	345000	630840	874320	785760	953520	
321 <sup>2</sup>	31 <sup>4</sup>	2 <sup>3</sup> 1	2 <sup>2</sup> 1 <sup>3</sup>	21 <sup>5</sup>	1 <sup>7</sup>				
1328160	1694400	1610640	2072160	2433600	2620800				
6	51	42	41 <sup>2</sup>	3 <sup>2</sup>	321	31 <sup>3</sup>	2 <sup>3</sup>	2 <sup>2</sup> 1 <sup>2</sup>	
261750	840300	1532250	2121660	1907400	3217080	4095696	3896460	5001672	
21 <sup>4</sup>	1 <sup>6</sup>								
5853744	6274080								
5	41	32	31 <sup>2</sup>	2 <sup>2</sup> 1	21 <sup>3</sup>	1 <sup>5</sup>			
1323500	3322300	5017400	6358480	7740360	8988720	9530400			
4	31	2 <sup>2</sup>	21 <sup>2</sup>	1 <sup>4</sup>					
3763625	7093100	8590950	9830340	10212600					
3	21	1 <sup>3</sup>							
6011500	8045700	8043000							
2	1 <sup>2</sup>	1							
4985500	4595000	1650000							

This conjecture has been checked for any  $K_r$  with  $r \leq 32$ . Starting from [10, Theorem 3.5.12], it may probably be proved by the method given in the next section.

We have also obtained the values of the positive numbers  $a_\rho^{(4)}$ . Listing them here would be tedious, but they are available upon request to the author.

## 6 Proof of Theorem 2

We start from the following lemma of symmetric function theory. It is better understood in the language of  $\lambda$ -rings. This method allows to handle symmetric functions acting on “sums”, “products” or “multiples” of alphabets. Here we shall not enter into details, and refer the reader to [7, Chapter 2] or [8, Section 3] for a short survey.

If  $f$  is a symmetric function, we denote  $f[\mathbf{A}]$  its  $\lambda$ -ring action on the alphabet  $\mathbf{A}$ , which should not be confused with its evaluation  $f(\mathbf{A})$ . For instance  $p_n[-z+2] = -z^n + 2$  and  $p_n(-z+2) = (-z+2)^n$ .

**Lemma 1.** *On any alphabet  $\mathbf{A}$  and for any positive integer  $n$ , we have*

$$\begin{aligned} \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} e_i e_j e_k &= \sum_{|\mu|=n} (-1)^{n-l(\mu)} 3^{l(\mu)} z_\mu^{-1} p_\mu, \\ \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} i^2 e_i e_j e_k &= \sum_{|\mu|=n} (-1)^{n-l(\mu)} 3^{l(\mu)-2} (n^2 + 2p_2(\mu)) z_\mu^{-1} p_\mu. \end{aligned}$$

*Sketch of proof.* Recall the “Cauchy formula” [7, (1.6.6)], or [9, (4.1) p. 62-65] or [8, p. 222],

$$e_n[\mathbf{AB}] = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z_\mu^{-1} p_\mu[\mathbf{A}] p_\mu[\mathbf{B}].$$

The first relation evaluates  $e_n[3\mathbf{A}]$  by using this formula together with the identity  $p_\mu[p] = p^{l(\mu)}$  valid for any real number  $p$ .

For the second relation, we evaluate similarly  $e_n[(z+2)\mathbf{A}]$ . Then we differentiate two times and fix  $z = 1$ . At the left-hand side we get  $\sum_{i+j+k=n} i(i-1)e_i e_j e_k[\mathbf{A}]$ . At the right-hand side, we compute

$$\partial_z^2 (p_\mu[z+2]) \Big|_{z=1} = \partial_z^2 \left( \prod_{i \geq 1} (z^i + 2)^{m_i(\mu)} \right) \Big|_{z=1} = 3^{l(\mu)-2} (n^2 - 3n + 2p_2(\mu)).$$

Observe that by differentiating  $r$  times, we might similarly get  $\sum_{i+j+k=n} \binom{i}{r} e_i e_j e_k$ .  $\square$

Specializing the alphabet  $\mathbf{A}$  as in Section 5, so that

$$Q_i = -p_i(\mathbf{A})/i, \quad C_i = (-1)^i e_i(\mathbf{A}),$$

we obtain

$$\sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} (a + bi + ci^2) C_i C_j C_k = \sum_{|\mu|=n} 3^{l(\mu)} \left( a + \frac{b}{3}n + \frac{c}{9}(n^2 + 2p_2(\mu)) \right) Q_\mu.$$

By insertion in the expression [5, Theorem 3.3]

$$K_{r,r-3} = \binom{r+1}{3} \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=r-3}} (a(r) + b(r)i^2) C_i C_j C_k,$$

we obtain

$$g_2(\mu) = a(r) + \frac{1}{9}b(r)((r-3)^2 + 2p_2(\mu)).$$

Since  $|\mu| = p_1(\mu) = r - 3$ , this can be rewritten

$$g_2 = \frac{1}{8640} \left( 8p_2p_1^2 + p_1^4 + 48p_2p_1 + 12p_1^3 + 60p_2 + 45p_1^2 + 54p_1 \right).$$

Using ACE [12] the conversion to monomial functions is performed immediately.  $\square$

## 7 Theorem 1 revisited

In Section 3 (proof of Theorem 1) we used the property

$$\sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} (a + bi + ci^2) C_i C_j C_k = \frac{1}{2} \sum_{|\mu|=n} (l(\mu) + 2)! \left( a + \frac{b}{3}n + \frac{c}{6}(n^2 + p_2(\mu)) \right) \mathcal{R}_\mu,$$

which may also be proved by  $\lambda$ -rings method. It is obtained by specialization of the following lemma.

**Lemma 2.** *On any alphabet  $\mathbf{A}$  and for any positive integer  $n$ , we have*

$$\begin{aligned} \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} e_i e_j e_k &= \frac{1}{2} \sum_{|\mu|=n} (-1)^{n-l(\mu)} \frac{(l(\mu) + 2)!}{\prod_i m_i(\mu)!} h_\mu, \\ \sum_{\substack{(i,j,k) \in \mathbb{N}^3 \\ i+j+k=n}} i^2 e_i e_j e_k &= \frac{1}{12} \sum_{|\mu|=n} (-1)^{n-l(\mu)} \frac{(l(\mu) + 2)!}{\prod_i m_i(\mu)!} (n^2 + p_2(\mu)) h_\mu. \end{aligned}$$

*Sketch of proof.* Recall the “Cauchy formula” [7, (1.6.3)], or [9, (4.2) p. 62-65] or [8, p. 222],

$$(-1)^n e_n[\mathbf{AB}] = \sum_{|\mu|=n} m_\mu[-\mathbf{B}] h_\mu[\mathbf{A}].$$

The first relation evaluates  $e_n[3\mathbf{A}]$  by using this formula and the identity [7, (2.2.2)] valid for any real number  $p$ ,

$$m_\mu[p] = p(p-1) \cdots (p-l(\mu)+1) / \prod_i m_i(\mu)!.$$

For the second relation, we evaluate similarly  $e_n[(z+2)\mathbf{A}]$ . Then we differentiate two times and fix  $z = 1$ , obtaining  $\sum_{i+j+k=n} i(i-1)e_i e_j e_k[\mathbf{A}]$  at the left-hand side. At the right-hand side, we compute

$$\prod_{i \geq 1} m_i(\mu)! \partial_z^2 (m_\mu[-z-2]) \Big|_{z=1} = (-1)^{l(\mu)} (l(\mu) + 2)! (n^2 - 2n + p_2(\mu)) / 12.$$

$\square$

## 8 Final remark

In this note, we have considered three conjectural developments of the Kerov component  $K_{r,r-2k+1}$ , namely up to  $\binom{r+1}{3}$ ,

$$\begin{aligned} \sum_{\substack{\nu \in \mathbb{N}^{2k-1} \\ |\nu|=r-2k+1}} F_k(\nu) \prod_{i=1}^{2k-1} C_{\nu_i} &= \sum_{|\mu|=r-2k+1} (l(\mu) + 2k - 2)! f_k(\mu) \mathcal{R}_\mu \\ &= \sum_{|\mu|=r-2k+1} (2k - 1)^{l(\mu)} g_k(\mu) \mathcal{Q}_\mu. \end{aligned}$$

As indicated above, these relations are better understood in the framework of symmetric functions. Choosing

$$(i - 1)R_i = -h_i(\mathbf{A}), \quad Q_i = -p_i(\mathbf{A})/i, \quad C_i = (-1)^i e_i(\mathbf{A}),$$

they are the specializations at  $\mathbf{A}$  of the abstract identities

$$\begin{aligned} \sum_{\substack{\nu \in \mathbb{N}^{2k-1} \\ |\nu|=n}} F_k(\nu) e_\nu &= \sum_{|\mu|=n} (-1)^{n-l(\mu)} f_k(\mu) \frac{(l(\mu) + 2k - 2)!}{\prod_i m_i(\mu)!} h_\mu \\ &= \sum_{|\mu|=n} (-1)^{n-l(\mu)} g_k(\mu) (2k - 1)^{l(\mu)} z_\mu^{-1} p_\mu. \end{aligned}$$

Moreover these identities are themselves related with the classical Cauchy formulas. Using the values of  $p_\mu[p]$  and  $m_\mu[p]$  given above, they may be written

$$\begin{aligned} (-1)^n \sum_{\substack{\nu \in \mathbb{N}^{2k-1} \\ |\nu|=n}} F_k(\nu) e_\nu &= \sum_{|\mu|=n} f_k(\mu) m_\mu[-2k + 1] h_\mu \\ &= \sum_{|\mu|=n} g_k(\mu) p_\mu[-2k + 1] z_\mu^{-1} p_\mu. \end{aligned}$$

Therefore it seems plausible that the conjectured positivity properties of  $f_k$  and  $g_k$  are equivalent, and reflect some abstract pattern of the theory of symmetric functions.

## References

- [1] P. Biane, *Representations of symmetric groups and free probability*, Adv. Math. **138** (1998), 126-181.
- [2] P. Biane, *Characters of symmetric groups and free cumulants*, Lecture Notes in Math. **1815** (2003), 185-200, Springer, Berlin, 2003.

- [3] P. Biane, *On the formula of Goulden and Rattan for Kerov polynomials*, Sémin. Lothar. Combin., **55** (2006), article B55d.
- [4] V. Féray, *Combinatorial interpretation and positivity of Kerov's character polynomials*, arXiv 0710.5885.
- [5] I. P. Goulden, A. Rattan, *An explicit form for Kerov's character polynomials*, Trans. Amer. Math. Soc. **359** (2007), 3669–3685.
- [6] S. V. Kerov, talk at IHP Conference (2000).
- [7] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS Regional Conference Series in Mathematics **99**, Amer. Math. Soc., Providence, 2003.
- [8] M. Lassalle, *Une  $q$ -spécialisation pour les fonctions symétriques monomiales*, Adv. Math. **162** (2001), 217–242.
- [9] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, second edition, Oxford, 1995.
- [10] A. Rattan, *Character polynomials and Lagrange inversion*, Thesis (2005), Waterloo University.
- [11] P. Śniady, *Asymptotics of characters of symmetric groups and free probability*, Discrete Math. **306** (2006), 624–665.
- [12] S. Veigneau, *ACE, an Algebraic Combinatorics Environment for the computer algebra system Maple*, available at <http://phalanstere.univ-mlv.fr/~ace/>